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# On the Goldie Quotient Ring of the Enveloping Algebra of a Classical Simple Lie Superalgebra[1]

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If $\mathfrak{g}$ is a classical simple Lie superalgebra ($\mathfrak{g} \neq P(n)$), the enveloping algebra $U(\mathfrak{g})$ is a prime ring and hence has a simple artinian ring of quotients $Q(U(\mathfrak{g}))$ by Goldie's Theorem. We show that if $\mathfrak{g}$ has Type I then $Q(U(\mathfrak{g}))$ is a matrix ring over $Q(U(\mathfrak{g}_0))$. On the other hand, if $\mathfrak{g} = osp(1, 2r)$ then by extending the center of $U(\mathfrak{g})$ we obtain a prime ring whose Goldie quotient ring is a matrix ring over the quotient division ring of a Weyl algebra. This is an analog of a result of Gelfand and Kirillov. © 2001 Academic Press

If $U$ is a prime Noetherian ring, we denote the Goldie quotient ring of $U$ by $Q(U)$. We work over an algebraically closed base field $K$ of characteristic zero. However, it will be necessary to consider field extensions of $K$ which are not algebraically closed. If $\mathfrak{g}$ is a classical simple Lie superalgebra, $\mathfrak{g} \neq P(n)$, then by a result of Bell [Be, Theorem 3.6] the enveloping algebra $U(\mathfrak{g})$ is prime. Hence $Q(U(\mathfrak{g})) \cong M_t(D)$ the algebra of all $t \times t$ matrices over a skew field $D$. Following [J2, 11.3] we call $t$ and $D$ the Goldie rank and the Goldie field of $U(\mathfrak{g})$.

If $\mathfrak{g}$ is an algebraic Lie algebra, then a well known conjecture of Gelfand and Kirillov states that $Q(U(\mathfrak{g}))$ is the quotient division ring of a Weyl algebra $A_n$ over a purely transcendental field extension of $K$. The conjecture is known to be false in general [AOV]. The conjecture is proved for $\mathfrak{g} = sl(n)$ in [GK1]. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a direct sum of Lie algebras, then $U(\mathfrak{g}) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$ has $Q(U(\mathfrak{g}_1)) \otimes Q(U(\mathfrak{g}_2))$ as a partial quotient ring. Using this observation, the conjecture can be demonstrated for

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direct sums of Lie algebras of type A. It is open for all other semisimple Lie algebras.

In this paper we study the Goldie quotient ring of $U(\mathfrak{g})$ and related algebras when $\mathfrak{g}$ is a classical simple Lie superalgebra. We make progress on this problem in two situations: namely when $\mathfrak{g}$ is a Lie superalgebra of Type I and when $\mathfrak{g} = osp(1, 2r)$. In the former case a rather easy argument shows that $Q(U(\mathfrak{g}))$ is a matrix algebra over $Q(U(\mathfrak{g}_0))$. Thus the description of $Q(U(\mathfrak{g}))$ is reduced to the analogous problem for semisimple Lie algebras.

Let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$, and $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h}^W)$ the Kac–Harish Chandra homomorphism. Perhaps the best result known when $\mathfrak{g}$ is an arbitrary semisimple Lie algebra is that $Q(U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} S(\mathfrak{h}))$ is isomorphic to $Q(A_n \otimes S(\mathfrak{h}))$ for some $n$ [GK2]. When $\mathfrak{g}$ is the Lie superalgebra $osp(1, 2r)$ we show that $Q(U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} S(\mathfrak{h}))$ is isomorphic to a matrix ring over $Q(A_n \otimes S(\mathfrak{h}))$. This is a consequence of a result of Gorelik and Lanzmann [GL] and some methods we develop to study prime and primitive factors of $U(\mathfrak{g})$ in general. As another application of these methods we describe $U(\mathfrak{g})/\operatorname{ann} \tilde{M}(\lambda)$ for $\lambda$ regular.

On the other hand it follows from [AL] that when $\mathfrak{g} = osp(1, 2r)$, $U(\mathfrak{g})$ is a domain and hence $Q(U(\mathfrak{g}))$ is a division ring. We show that if $\mathfrak{g} = osp(1, 2)$ then $Q(U(\mathfrak{g}))$ is not the division ring of a Weyl algebra, and it is likely that this is true also for $r > 1$. Thus we have some examples of division rings which are not Weyl fields but which become matrix algebras over Weyl fields after extending the center.

It is worth remarking that the enveloping algebras of Lie superalgebras of Type I and of the Lie superalgebras $osp(1, 2r)$ are much better understood than those of other classical simple Lie superalgebras.

For simplicity we assume throughout that $\mathfrak{g} \neq Q(n)$. Then a subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$ is actually a Cartan subalgebra of $\mathfrak{g}$.

*Notation.* For $\lambda \in \mathfrak{h}^*$ we denote by $\tilde{M}(\lambda)$ and $M(\lambda)$ the Verma modules over $U(\mathfrak{g})$ and $U(\mathfrak{g}_0)$ with highest weight $\lambda$, respectively [M1, Sect. 1]. The unique graded-simple factor modules of $\tilde{M}(\lambda)$ and $M(\lambda)$ are denoted by $\tilde{L}(\lambda)$ and $L(\lambda)$.

## 1. TYPE I BASIC CLASSICAL SIMPLE LIE SUPERALGEBRAS

1.1. A classical simple Lie superalgebra $\mathfrak{g}$ is said to be of Type I if $\mathfrak{g} = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, a direct sum of two simple $\mathfrak{g}_0$-submodules. The Type I Lie superalgebras consist of the series $A(m, n)$, $C(n)$, and $P(n)$. By checking each case, or by using [Sch, Proposition 3, p. 96] we have that $[\mathfrak{g}_1^+, \mathfrak{g}_1^+] = [\mathfrak{g}_1^-, \mathfrak{g}_1^-] = 0$.

Let $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$ be a triangular decomposition of $\mathfrak{g}_0$ and set

$$\mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \mathfrak{g}_1^+, \qquad \mathfrak{n}^- = \mathfrak{n}_0^- \oplus \mathfrak{g}_1^-$$

and

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+, \qquad \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1^+.$$

Note that the definition of the Verma modules $\tilde{M}(\lambda)$ depends on which $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$ we call $\mathfrak{g}_1^+$. We say a Type I Lie superalgebra $\mathfrak{g}$ is basic if $\mathfrak{g} \neq P(n)$.

LEMMA. (a) *$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h}^+ \oplus \mathfrak{n}^+$ is a triangular decomposition of $\mathfrak{g}$.*

(b) *$I = \mathfrak{g}_1^+ U(\mathfrak{p}) = U(\mathfrak{p})\mathfrak{g}_1^+$ is a nilpotent ideal of $U(\mathfrak{p})$ such that $U(\mathfrak{p})/I \cong U(\mathfrak{g}_0)$.*

1.2. LEMMA. (a) *Let $R \subseteq S$ be a ring extension such that $S$ is a free of rank $t$ as a left $R$-module. Let $Q$ be an ideal of $R$, and $J_Q = \operatorname{ann}_S(S/SQ)$.*

(a) *Then there is a ring homomorphism $\phi_Q : S \to M_t(R/Q)$ with kernel $J_Q$,*

(b) *if $N$ is a left $R$-module with annihilator $Q$, then $J_Q = \operatorname{ann}_S(S \otimes_R N)$.*

*Proof.* (a) Let $x_1, \dots, x_t$ be a basis for $S$ as a left $R$-module. For $s \in S$, there exist unique elements $\phi(s)_{ji} \in R$ such that

$$sx_i = \sum_{j=1}^{t} x_j \phi(s)_{ji}.$$

Clearly the map $\phi : S \to M_t(R)$ sending $s$ to $(\phi(s)_{ji})$ is a homomorphism. Composing $\phi$ with the natural map $M_t(R) \to M_t(R/Q)$ we obtain $\phi_Q$. Finally, we have

$$\begin{aligned} \operatorname{Ker} \phi_Q &= \{ s \in S \mid \phi(s)_{ji} \in Q \text{ for all } i, j \} \\ &= J_Q. \end{aligned}$$

(b) This follows from [BGR, Lemma 10.4].

1.3. Let $\mathfrak{g}$ be a basic classical simple Lie superalgebra of Type I. We identify the Goldie quotient ring of $U(\mathfrak{g})$.

THEOREM. *Let $\mathfrak{g}$ be a simple Lie superalgebra of Type $A(m, n)$ or $C(n)$, and let $r = \dim \mathfrak{g}_1^+$ and $t = 2^r$. Then*

$$Q(U(\mathfrak{g})) \cong M_t(Q(U(\mathfrak{g}_0))).$$

*In particular the Goldie rank of $U(\mathfrak{g})$ is equal to $2^r$ and the Goldie fields of $U(\mathfrak{g})$ and $U(\mathfrak{g}_0)$ are isomorphic.*

*Proof.* Let $R = U(\mathfrak{p})$, $S = U(\mathfrak{g})$, and $I = \mathfrak{g}_1^+ U(\mathfrak{p})$ as in Lemma 1.1. Note that $S$ is a free left $R$-module of rank $t$. We set $\psi = \phi_I$ as in Lemma 1.2. As noted in [M1, 3.4], $\tilde{M}(\lambda) \cong U(\mathfrak{g}) \otimes_{U(p)} M(\lambda)$, so if $s \in \operatorname{Ker} \psi$, then $s \in \operatorname{ann}_S \tilde{M}(\lambda)$ for all $\lambda \in \mathfrak{h}^*$. Hence by [LM, Theorem C], $s \in \bigcap_{\lambda \in \mathfrak{h}^*} \operatorname{ann}_S \tilde{L}(\lambda) = 0$.

Thus the image of $\psi$ is a free $U(\mathfrak{g}_0)$-module of rank $t^2$. However, the same is true for the matrix ring $M_t(U(\mathfrak{g}_0))$. Hence the image of $\psi$ has the same Goldie quotient ring as $M_t(U(\mathfrak{g}_0))$.

## 2. THE LIE SUPERALGEBRAS $osp(1, 2r)$

2.1. Let $\mathfrak{g} = osp(1,2)$, $C$ the Casimir element of $U(\mathfrak{g})$ as in [Pi], $\Omega = C + 1/16$, and $P_\lambda = (\Omega - \lambda)$ for $\lambda \neq 0$. The proof of the following facts can be found in [Pi].

LEMMA. (a) $Z(\mathfrak{g}) = K[\Omega]$.

(b) *There exists $T \in U(\mathfrak{g})$ such that $P_0 = U(\mathfrak{g})T = TU(\mathfrak{g})$ is a primitive ideal. Also $P_0^2 = \Omega U(\mathfrak{g})$.*

(c) *The minimal nonzero prime ideals of $U(\mathfrak{g})$ are the ideals $P_\lambda$, $\lambda \in K$ and all these ideals are primitive. Also $P_\lambda$ is completely prime if and only if $\lambda = 0$.*

COROLLARY. *If $\mathfrak{g} = osp(1,2)$, then $Q = Q(U(\mathfrak{g}))$ has center $K(\Omega)$.*

*Proof.* Suppose $z$ is central in $Q$ and set

$$I = \{a \in U(\mathfrak{g}) \mid az \in U(\mathfrak{g})\}.$$

Since $I$ is a nonzero ideal of $U(\mathfrak{g})$, $I$ contains a product of prime ideals. Thus by the Lemma, $I$ contains a product of the ideals $P_\lambda$, and we can assume that $P_0$ occurs with even exponent in this product. Thus $I \cap Z(\mathfrak{g}) \neq 0$, and the result follows.

2.2. THEOREM. *If $\mathfrak{g} = osp(1,2)$, then $Q(U(\mathfrak{g}))$ is not isomorphic to $Q(A_n \otimes_K C)$ for any commutative $K$-algebra $C$.*

*Proof.* Let $D_n = Q(A_n)$, and let $\Omega$ be the Casimir element of $U(\mathfrak{g})$. By Corollary 2.1 it suffices to show that there cannot exist an isomorphism

$$\phi : Q(U(\mathfrak{g})) \to Q(A_n \otimes K[t]) = D_n(t)$$

such that $\phi(\Omega) = t$.

For $\lambda \in K$, let $\mathscr{C}(\lambda)$ be the set of elements in $D_n[t]$ which are regular modulo the ideal $(t - \lambda)$. Then $\mathscr{C}(\lambda)$ is Ore in $D_n[t]$, and we can define a homomorphism

$$\sigma_\lambda : D_n[t]_{\mathscr{C}(\lambda)} \to D_n$$

such that $\sigma_\lambda(t) = \lambda$ and $\sigma_\lambda$ restricts to the identity on $D_n$.

If $f = \sum a_i t^i \in D_n[t]$, $f \neq 0$, then using Vandermonde determinants, we see that there is a finite subset $\Lambda_1$ of $K$ such that $f(\lambda) = \sum a_i \lambda^i \neq 0$ for all $\lambda \in \Lambda'_1 = K \setminus \Lambda_1$. Thus since $U(\mathfrak{g})$ is a finitely generated algebra, there is a finite subset $\Lambda_2$ of $K$ such that $\phi(U(\mathfrak{g})) \subseteq D_n[t]_{\mathscr{C}(\lambda)}$ for all $\lambda \in \Lambda'_2$.

Now for $\lambda \in \Lambda'_2$, let $\psi_\lambda = \sigma_\lambda \circ \phi$. Then $J_\lambda = \operatorname{Ker} \psi_\lambda$ is a completely prime primitive ideal of $U(\mathfrak{g})$ with $\Omega - \lambda \in J_\lambda$. This contradicts Lemma 2.1.

2.3. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be classical simple, and $M, N$ $\mathbb{Z}_2$-graded $\mathfrak{g}$-modules. Recall that $\operatorname{Hom}_K(M, N)$ becomes a $\mathfrak{g}$-module when we set

$$(x.f)(m) = xf(m) - (-1)^{\bar{x}\bar{f}} f(xm),$$

where $\bar{x}, \bar{f}$ denote the degree in $\mathbb{Z}_2$ of $x, f$, respectively. We define

$$\mathscr{L}(M, N) = \{ f \in \operatorname{Hom}_K(M, N) \mid \dim U(\mathfrak{g}).f < \infty \}.$$

Note that since $U(\mathfrak{g})$ is finitely generated as a $U(\mathfrak{g}_0)$-module, $\mathscr{L}(M, N)$ is equal to the set of elements of $\operatorname{Hom}_K(M, N)$ which are $\mathfrak{g}_0$-finite as in [J2, 6.8].

Now suppose that $M = \oplus_{i \in I} M_i$ is a direct sum of $\mathfrak{g}_0$-modules. It is clear that there is an isomorphism

$$\mathscr{L}(M, M) \cong \oplus_{i, j \in I} \mathscr{L}(M_i, M_j)$$

induced by the inclusion and projection maps. Furthermore

$$\mathscr{L}(M_j, M_k) \mathscr{L}(M_i, M_j) \subseteq \mathscr{L}(M_i, M_k)$$

for $i, j, k \in I$. Later we shall pass to the opposite ring for the convenience of having multiplication in $\mathscr{L}(M, M)$ resemble matrix multiplication.

2.4. The next result will help us recognize a matrix ring over a division ring.

LEMMA. *Let $T$ be a ring such that*

(1) $T = \oplus_{1 \leq i, j \leq N} T_{ij}$ *as an abelian group,* $D^{(i)} = T_{ii}$ *is a division ring for all* $i$, *and* $T_{ij} \neq 0$ *for all* $i, j$.

(2) $T_{ij}T_{jk} \subseteq T_{ik}$ *and* $T_{ij}T_{k\ell} = 0$ *if* $j \neq k$.

(3) *If* $x \in T_{ij}$, $y \in T_{jk}$, $x \neq 0 \neq y$, *then* $xy \neq 0$.

*Then* $D^{(i)} \cong D^{(j)}$ *for all* $i$, $j$, *and* $T \cong M_N(D^{(i)})$.

*Proof.* The hypotheses imply that $T_{ij}$ has dimension one as a left vector space over $D^{(i)}$ and as a right vector space over $D^{(j)}$. Indeed if $m_1, m_2$ are nonzero elements of $T_{ij}$ and $n \in T_{ji}$, $n \neq 0$, there exists $s_1, s_2 \in T_{ii}$ such that $s_1 m_1 n = s_2 m_2 n$ and this implies $s_1 m_1 - s_2 m_2 = 0$. Since $T_{ij}T_{jk}$ is a nonzero subspace of $T_{ik}$ it follows that

$$T_{ij}T_{jk} = T_{ik}$$

for all $i, j, k$. If $e_{ii}$ is the identity element of $T_{ii}$ there exists $e_{1i} \in T_{1i}$ and $e_{i1} \in T_{i1}$ such that $e_{1i}e_{i1} = e_{11}$. The fact that $T_{1i}$ has dimension one over $D^{(i)}$ ensures that we can write $e_{11}$ as a product of elements from $T_{1i}T_{i1}$ and not just as a sum of products. Since $e_{i1}e_{1i}$ is a nonzero idempotent in $T_{ii}$, we have $e_{i1}e_{1i} = e_{ii}$. The conditions in (2), (3) imply that $T_{ij}$ is a unital module over $D^{(i)}$ and $D^{(j)}$, that is, $e_{ii}x = x = xe_{jj}$ for all $x \in T_{ij}$. Now set $e_{ij} = e_{i1}e_{1j}$. It is easily checked that $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$. Thus by [Pa, Lemma 6.1.5] we have the result.

2.5. Next we study $\mathscr{L}(M, M)$ when $M$ is a direct sum of Verma modules over $U(\mathfrak{g}_0)$. First define $P_0, P_0^+$ as in [M3]. Then $P_0$ is the analog of the weight lattice for the reductive Lie algebra $\mathfrak{g}_0$, and $P_0^+$ is a subsemigroup of $P_0$ satisfying $\lambda \in P_0^+$ if and only if $\dim L(\lambda) < \infty$. If $\Lambda \in \mathfrak{h}^*/P_0$ is a coset of $P_0$ we define $\Lambda^+$, $\Lambda^{++}$, $\Lambda_0^+$, and $\Lambda_0^{++}$ as in [M3, 0.5]. Fix $\Lambda \in \mathfrak{h}^*/P_0$ and suppose that $M = \oplus_{i=1}^N M_i$ where each $M_i = M(\lambda_i)$ with $\lambda_i \in \Lambda^+$ is a Verma module for $U(\mathfrak{g}_0)$. If $F$ is a field extension of $K$ then $D_n(F)$ denotes the quotient division ring of the $n$th Weyl algebra $A_n(F)$ over $F$. Set $D_n = D_n(K)$.

THEOREM. *Let $n$ be the number of positive roots of $\mathfrak{g}_0$. Then $\mathscr{L}(M, M)$ is a prime Noetherian ring with Goldie quotient ring $M_N(D_n)$.*

*Proof.* Since $\lambda_i - \lambda_j \in P_0$ we have $\mathscr{L}(M_i, M_j) \neq 0$ for all $i, j$ by [J2, 6.9 (7)]. In addition if $U_i = \mathscr{L}(M_i, M_i)$, then $U_i$ is a homomorphic image of $U(\mathfrak{g})$ by [J2, 6.9 (8)]. This hypothesis is necessary to apply the results of [J2, 11.9–11.12]. Note that $\mathscr{L}(M_i, M_j)$ is a $U_j - U_i$-bimodule. Recall that a $U(\mathfrak{g}_0)$-module is homogeneous if it has the same Gelfand–Kirillov dimension as any nonzero submodule. Since each $M_i$ is homogeneous by [J2, 8.11(1) and 9.1(3)] it follows from [J2, Lemma 10.12 and Bemerkung 10.2] that each $\mathscr{L}(M_i, M_j)$ is homogeneous both as a left $U_j$-module and as a right $U_i$-module.

By [C, Théorème 10.4], $U_i$ is a Noetherian domain whose Goldie quotient ring is isomorphic to $D_n$. If $\mathscr{C}_i$ is the set of regular elements of $U_i$ we can view $\mathscr{C} = \Pi \mathscr{C}_i$ as the set of regular elements of the subring $\oplus_i U_i$ of $\mathscr{L}(M, M)$.

To finish the proof we need the following lemma.

LEMMA. *The set $\mathscr{C}$ is an Ore set of regular elements in $\mathscr{L}(M, M)$.*

*Proof.* Suppose $c = \Pi_k c_k \in \mathscr{C}$ and $r = \sum_{i,j} r_{ij} \in \mathscr{L}(M, M)$ with $c_k \in \mathscr{C}_k$ and $r_{ij} \in \mathscr{L}(M_i, M_j)$. Then

$$cr = \sum_{i,j,k} c_k r_{ij} \delta_{jk} = \sum_{i,j} c_j r_{ij}$$

with $c_j r_{ij} \in \mathscr{L}(M_i, M_j)$. Thus $cr = 0$ implies that $c_j r_{ij} = 0$ for all $i, j$ and so by [J2, Lemma 11.9], $r_{ij} = 0$ for all $i, j$ and $r = 0$.

In addition, by [J2, Satz 11.12], we have

$$\mathscr{L}(M_i, M_j)\mathscr{C}_i^{-1} = \mathscr{C}_j^{-1}\mathscr{L}(M_i, M_j).$$

Thus with $i, j$ fixed, there exists $d_j^{(i)} \in \mathscr{C}_j$ and $r'_{ij} \in \mathscr{L}(M_i, M_j)$ such that

$$d_j^{(i)} r_{ij} = r'_{ij} c_i.$$

Also the elements $d_j^{(i)}$, as $i$ varies, have a common left multiple $d_j \in \mathscr{C}_j$ and hence there exist $r''_{ij} \in \mathscr{L}(M_i, M_j)$ such that

$$d_j r_{ij} = r''_{ij} c_i$$

for all $i$ and $j$. If $d = \Pi_k d_k$ and $r'' = \sum_{i,j} r''_{ij}$ then $dr = r''c$. This shows the left Ore condition, and the right Ore condition for $\mathscr{C}$ is shown similarly.

It is now easy to conclude the proof of the theorem. Let $T = (\mathscr{C}^{-1}\mathscr{L}(M, M))^{op}$ and $T_{ij} = (\mathscr{C}_j^{-1}\mathscr{L}(M_i, M_j))^{op}$. We have $T_{ii} \cong D_n^{op} \cong D_n = Fract(U_i)$ as already mentioned. It is easy to see that hypotheses (1), (2) of Lemma 2.4 are satisfied. Hypothesis (3) is satisfied by [C, Lemma 10.3 (ii)]. Hence $\mathscr{C}^{-1}\mathscr{L}(M, M) \cong T^{op} \cong M_N(D_n)$ as required.

2.6. Theorem 2.5 applies to the algebras $\mathscr{L}(\tilde{M}(\lambda), \tilde{M}(\lambda))$ under quite general conditions. Let $\Gamma$ be the set of sums of distinct odd positive roots, and let $\kappa : \Gamma \to \mathbb{N}$ be the function defined by

$$\Pi_{\alpha \in \Delta_1^+}(1 + e^{-\alpha}) = \sum_{\gamma \in \Gamma} \kappa(\gamma) e^{-\gamma}.$$

Fix $\Lambda \in \mathfrak{h}^*/P_0$ and $\lambda \in \mathfrak{h}^*$ such that $\lambda - \gamma \in \Lambda_0^+$ for all $\gamma \in \Gamma$, where $\Lambda = \lambda + P_0$. This condition will hold for any classical simple Lie superalgebra, provided we choose $\lambda$ sufficiently dominant. By [M2, Theorem 3.2], $\tilde{M}(\lambda)$ has a finite filtration as a $\mathfrak{g}_0$-module with factors of the form $M(\lambda-\gamma)$ where $\gamma \in \Gamma$. In addition the module $M(\lambda-\gamma)$ occurs with multiplicity $\kappa(\gamma)$ in this filtration. Now under the stated hypothesis on $\lambda$, each module $M(\lambda-\gamma)$ is projective in the category $\mathscr{O}$ by [J2, Lemma 4.8].

Hence as a $\mathfrak{g}_0$-module

$$\tilde{M}(\lambda) = \oplus_{\gamma \in \Gamma} M(\lambda-\gamma)^{\kappa(\gamma)}.$$

COROLLARY. *Under the above hypothesis* $\mathscr{L}(\tilde{M}(\lambda), \tilde{M}(\lambda))$ *is a prime Noetherian ring with*

$$Q\big(\mathscr{L}\big(\tilde{M}(\lambda), \tilde{M}(\lambda)\big) \cong M_N(D_n),$$

*where* $N = 2^{|\Delta_1^+|}$ *and* $n = |\Delta_0^+|$.

2.7. Following [J1, Chap. 4] we make some remarks about Verma modules defined over $K$-algebras. If $V$ is a vector space over $K$ and $L$ is a commutative $K$-algebra, we set $V_L = V \otimes_K L$. We warn the reader that the notation $V_L$ is also used here and in [J1] to refer to $L$-modules which do not arise in this way. We embed $V$ into $V_L$ as $V \otimes 1$. When $V$ carries an additional structure, such as that of an associative or Lie algebra, then $V_L$ carries the corresponding structure. Let $\mathfrak{g}$ be a classical simple Lie superalgebra. If $\lambda \in \mathfrak{h}_L^* = \mathrm{Hom}_L(\mathfrak{h}_L, L)$ we denote by $L_\lambda$ the free $L$-module of rank one which is made into a $\mathfrak{b}_L$-module by allowing $\mathfrak{n}_L$ to act trivially and $h$ to act as multiplication by $\lambda(h)$ for $h \in \mathfrak{h}_L$. Then we form the Verma module $\tilde{M}(\lambda)_L = U(\mathfrak{g}_L) \otimes_{U(\mathfrak{b}_L)} L_\lambda$. Note that if $\lambda \in \mathfrak{h}^*$ then $\tilde{M}(\lambda)_L \cong \tilde{M}(\lambda) \otimes L$ so this notation is consistent with that used earlier.

We now specialize to the case $\mathfrak{g} = osp(1, 2r)$. The result below extends a result of Gorelik and Lanzmann [GL] when $L$ is an algebraically closed field.

LEMMA. *Suppose that $L$ is a domain and* $(\lambda+\rho, \alpha) \neq 0$ *for all odd roots* $\alpha$. *Then* $\mathrm{ann}_{U(\mathfrak{g})_L} \tilde{M}(\lambda)_L$ *is generated by its intersection with the center of* $U(\mathfrak{g})_L$.

*Proof.* To simplify the notation set $U_L = U(\mathfrak{g})_L$, $M_L = \tilde{M}(\lambda)_L$, $J_L = \mathrm{ann}_{U_L}(M_L)$, $Z_L = Z(U_L)$, and $I_L = J_L \cap Z_L$. Let $F$ be the field of fractions of L and $\overline{F}$ the algebraic closure of F. Then we have

$$\begin{aligned} J_L &= U_L \cap J_{\overline{F}} \\ &= U_L \cap U_{\overline{F}} I_{\overline{F}} \\ &= U_L \cap U_F I_F \\ &= U_L I_L. \end{aligned}$$

The second equality follows from [GL]. For the last equality it suffices to show that $U_L/U_L I_L$ is torsion-free as an $L$-module by [GW, Theorem 9.17]. By [M2, Theorem 1.5] there is a subspace $H$ of $U(\mathfrak{g})$ such that $U(\mathfrak{g}) = Z(\mathfrak{g}) \otimes H$. Hence $U_L = (Z(\mathfrak{g})_L) \otimes_L H_L$, and so $U_L/U_L I_L \cong H_L$ a free $L$-module. The remaining equalities are easily verified.

2.8. Fix $\Lambda \in \mathfrak{h}^*/P_0$. If $\lambda \in \Lambda^{++}$, then by [M3, Corollary 3.9], $\lambda - \gamma \in \Lambda_0^+$ for all $\gamma \in \Gamma$ so Corollary 2.6 applies to $U_\lambda = \mathscr{L}(\tilde{M}(\lambda), \tilde{M}(\lambda))$. In addition by [M2, Theorem 1.5] there is an ad-invariant subspace $H$ of $U(\mathfrak{g})$ such that $U(\mathfrak{g}) \cong H \otimes Z(\mathfrak{g})$ as ad $\mathfrak{g}$-modules. Also, if $E$ is any finite dimensional $U(\mathfrak{g})$-module, the multiplicity of $E$ as a composition factor of $H$ is equal to the dimension $\dim E^0$ of the zero weight space of $E$.

The condition $\lambda \in \Lambda^{++}$ implies that $(w.\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \Delta_1$. Thus by [GL], ann $\tilde{M}(w.\lambda)$ is generated by its intersection with $Z(\mathfrak{g})$. Hence we can identify $H$ with $U(\mathfrak{g})/\text{ann } \tilde{M}(w.\lambda)$ as an ad $\mathfrak{g}$-module. Now $U(\mathfrak{g})/\text{ann } \tilde{M}(w.\lambda) = U(\mathfrak{g})/\text{ann } \tilde{M}(\lambda)$ embeds in $U_\lambda$. Also the multiplicity of a finite dimensional $U(\mathfrak{g})$ module $E$ in $U_\lambda$ equals $\dim E^0$, by adapting the usual argument leading to [J2, 6.9(7)], so we obtain $U(\mathfrak{g})/\text{ann } \tilde{M}(w.\lambda) \cong U_\lambda$. In summary we have proved

THEOREM. *Suppose* $\mathfrak{g} = osp(1, 2r)$, $\lambda \in \Lambda^{++}$ *and* $w \in W$. *Then the inclusion*

$$U(\mathfrak{g})/\text{ann } \tilde{M}(w.\lambda) \hookrightarrow \mathscr{L}\big(\tilde{M}(\lambda), \tilde{M}(\lambda)\big) = U_\lambda$$

*is an isomorphism, and* $Q(U_\lambda) \cong M_N(D_n)$, *where* $N = 2^r$ *and* $n = r^2$.

Observe that the above proof works over an arbitrary field extension of K, since it only depends on a computation of the multiplicities of the finite dimensional simple $\mathfrak{g}$-modules.

2.9. We assume that $\mathfrak{g} = osp(1, 2r)$ and let $F = Fract(S(\mathfrak{h}))$, $N = 2^r$ and $n = r^2$. We regard $S(\mathfrak{h})$ as a $Z(\mathfrak{g})$-module via the Harish-Chandra homomorphism $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h})$. Then $U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} S(\mathfrak{h})$ is a $K$-algebra whose center is isomorphic to $S(\mathfrak{h})$.

THEOREM. *With the above notation we have*

$$Q\big(U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} S(\mathfrak{h})\big) \cong M_N(D_n(F)).$$

*Proof.* Let $\lambda \in \mathfrak{h}_F^*$ be the $F$-linear map defined by $(\lambda + \rho)(h) = h$ for $h \in \mathfrak{h}$. Let $M$ be the Verma module over $U(\mathfrak{g})_F$ with highest weight $\lambda$.

Note that if $z \in Z(\mathfrak{g})$ then $z$ acts on $M$ as the scalar $\psi(z)(\lambda + \rho) = \psi(z) \in F$. Thus by Lemma 2.7, $J = \operatorname{ann}_{U_F} M$ is the ideal of $U_F$ generated by elements of the form

$$z \otimes 1 - 1 \otimes \psi(z)$$

for $z \in Z(\mathfrak{g})$.

It is easy to see that $I = J \cap U(\mathfrak{g})_{S(\mathfrak{h})}$ is the kernel of the natural surjection from $U(\mathfrak{g}) \otimes_K S(\mathfrak{h})$ to $U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} S(\mathfrak{h})$. Thus

$$U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} S(\mathfrak{h}) \cong \left(U(\mathfrak{g}) \otimes_K S(\mathfrak{h})\right)/I$$

embeds in $U(\mathfrak{g})_F/J$ which is isomorphic to $\mathscr{L}(M, M)$ by Theorem 2.8. Since the image of a nonzero element of $S(\mathfrak{h})$ is regular in $\mathscr{L}(M, M)$, it follows that $\mathscr{L}(M, M)$ is isomorphic to the localization of $U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} S(\mathfrak{h})$ at the set of regular central elements. Now the result follows from Theorem 2.8.

## REFERENCES


[AL] M. Aubry and J. M. Lemaire, Zero divisors in enveloping algebras of graded Lie algebras, *J. Pure Appl. Algebra* **38** (1985), 159–166.

[AOV] J. Alev, A. Ooms, and M. Van den Bergh, A class of counterexamples to the Gelfand–Kirillov conjecture, *Trans. Amer. Math. Soc.* **348** (1996), 1709–1716.

[Be] A. Bell, A criterion for primeness of enveloping algebras of Lie superalgebras, *J. Pure Appl. Algebra* **69** (1990), 111–120.

[BGR] W. Borho, P. Gabriel, and R. Rentschler, "Primideale in Einhüllenden auflösbarer Lie-Algebren," Lecture Notes in Mathematics, Vol. 357, Springer-Verlag, Berlin, 1973.

[C] N. Conze, Algèbres d'opérateurs différentiels et quotients des algèbres enveloppantes, *Bull. Soc. Math. France* **102** (1974), 379–415.

[GK1] I. M. Gelfand and A. A. Kirillov, Sur les corps liés aux algèbres enveloppantes des algèbres de Lie, *Inst. Hautes Études Sci. Publ. Math.* **31** (1966), 5–19.

[GK2] I. M. Gelfand and A. A. Kirillov, The structure of the Lie field connected with a split semisimple Lie algebra, *Funct. Anal. Appl.* **3** (1969), 6–21.

[GL] M. Gorelik and E. Lanzmann, The annihilation theorem for the completely reducible Lie superalgebras, *Invent. Math.* **137** (1999), 651–680.

[GW] K. R. Goodearl and R. B. Warfield, Jr., "An Introduction to Noncommutative Noetherian Rings," London Math. Soc. Stud. Texts, Vol. 16, Cambridge Univ. Press, Cambridge, UK, 1989.

[J1] J. C. Jantzen, "Moduln mit einem höchsten Gewicht," Lecture Notes in Mathematics, Vol. 750, Springer-Verlag, Berlin, 1979.

[J2] J. C. Jantzen, Einhüllende Algebren halbeinfacher Lie-Algebren, Ergeb. Math. Grenzgeb., 3. Folge, Band 3, Springer-Verlag, Berlin, 1983.

[Kac] V. Kac, Representations of classical Lie superalgebras, *in* Lecture Notes in Mathematics, Vol. 676, pp. 579–626, Springer-Verlag, Berlin, 1977.

[LM] E. S. Letzter and I. M. Musson, Complete sets of representations of classical Lie superalgebras, *Lett. Math. Phys.* **31** (1994), 247–253.

[M1] I. M. Musson, A classification of primitive ideals in the enveloping algebra of a classical simple Lie superalgebra, *Adv. Math.* **91** (1992), 252–268.

[M2] I. M. Musson, On the center of the enveloping algebra of a classical simple Lie superalgebra, *J. Algebra* **193** (1997), 75–101.

[M3] I. M. Musson, The enveloping algebra of the Lie superalgebra $osp(1, 2r)$, *Representation Theory* **1** (1997), 405–423.

[Pa] D. S. Passman, "The Algebraic Structure of Group Rings," Wiley–Interscience, New York, 1977.

[Pi] G. Pinczon, The enveloping algebra of the Lie superalgebra $osp(1, 2)$, *J. Algebra* **132** (1990), 219–242.

[Sch] M. Scheunert, "The Theory of Lie Superalgebras," Lecture Notes in Mathematics, Vol. 716, Springer-Verlag, Berlin, 1979.